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Absolute weak C-embedding in Hausdorff spaces ☆

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Abstract

Answering a problem of A.V. Arhangel'skiĭ, we prove the following theorem: a Hausdorff space Y is weakly C-embedded in every larger Hausdorff space X if and only if either Y is compact or every real-valued continuous function on Y is constant.

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1. Introduction and results

All spaces are assumed to be T_1-spaces. A subspace Y of a space X is said to be C (respectively C^*)-*embedded* in X if every real-valued (respectively every bounded real-valued) continuous function on Y can be continuously extended over X.

In the study of relative normality, Arhangel'skiĭ says a subspace Y of a space X *weakly C-embedded* in X [2] if every real-valued continuous function on Y can be extended to a real-valued function on X which is continuous at each point of Y. It is obvious that C-embedding implies weak C-embedding. In fact, weak C-embedding is strictly weaker than z-embedding [14], where a subspace Y of a space X is said to be z-*embedded* in X if for every zero-set Z of Y there exists a zero-set Z' of X such that $Z' \cap Y = Z$.

Weak C-embedding plays an important role not only in the theory of relative topological properties but also in the extension theory of continuous functions. For classical results related to absolute embedding of continuous functions in the realm of Tychonoff spaces,

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recall the following two facts. A Tychonoff space Y is said to be *almost compact* if $|\beta Y - Y| \leqslant 1$, where βY denotes the Stone–Čech compactification of Y. Note that on a Tychonoff space Y, Y is almost compact if and only if for every two disjoint zero-sets of Y at least one of them is compact [1,9,11].

Fact 1 (Doss [8], Hewitt [13], Smirnov [16]; see also [1,11]). *Let Y be a Tychonoff space. Then, Y is C-embedded (or C^*-embedded) in every larger Tychonoff space X if and only if Y is almost compact.*

Fact 2 (Blair [6], Blair–Hager [7], Hager–Johnson [12]). *Let Y be a Tychonoff space. Then, Y is z-embedded in every larger Tychonoff space X if and only if Y is almost compact or Lindelöf.*

A corresponding result for weak C-embedding was recently obtained by Bella–Yaschenko [5] as follows;

Theorem 1.1 (Bella–Yaschenko [5]). *Let Y be a Tychonoff space. Then, Y is weakly C-embedded in every larger Tychonoff space X if and only if Y is almost compact or Lindelöf.*

On the other hand, Arhangel'skiĭ posed in [3, Problem 3.14] the following problem which motivates us to consider absolute weak C-embedding in the realm of Hausdorff spaces.

Problem (*Arhangel'skiĭ* [3]). When is a Hausdorff (Tychonoff) space Y weakly C-embedded in every larger Hausdorff space X?

In this paper, we give a solution to this problem as follows:

Theorem 1.2. *Let Y be a Hausdorff space. Then, Y is weakly C-embedded in every larger Hausdorff space X if and only if either Y is compact or every real-valued continuous function on Y is constant.*

For results in other classes of spaces, see Theorems 2.2 and 2.3. In Section 3, we give some remarks on absolute weak P-embedding in the realm of Hausdorff spaces. Undefined terminology may be found in [9].

2. Proof

First recall from [14] a characterization of weak C-embedding.

Lemma 2.1 [14]. *A subspace Y of a space X is weakly C-embedded in X if and only if every two disjoint zero-sets of Y can be put inside disjoint open subsets of X.*

Proof of Theorem 1.2. The "if" part is easy to see. Indeed, if Y is compact, the characterization of weak C-embedding in Lemma 2.1 works well. The other case is obvious.

To prove the "only if" part, assume that Y is weakly C-embedded in every larger Hausdorff space X. To prove either Y is compact or every real-valued continuous function on Y is constant, assume on the contrary that Y is non-compact and admits a non-constant continuous function $f : Y \to \mathbb{R}$. We may assume that $f : Y \to [0, 1]$ and $f(y_0) = 0$ and $f(y_1) = 1$ for some distinct points $y_0, y_1 \in Y$. Since Y is non-compact, we may assume that $f^{-1}([1/2, 1])$ is non-compact and put $A = f^{-1}([1/2, 1])$. Then, there is a collection $\mathcal{F}$ of non-empty closed subsets of A which is closed under finite intersections and $\bigcap \mathcal{F} = \emptyset$. Consider the set $X = Y \times (\omega + 1)$ with the topology induced as the following:

(i) Any point of $Y \times \omega$ is isolated.
(ii) For $x \in Y - \{y_0\}$, take as a neighborhood base at (x, ω) the sets $U \times (n, \omega]$, where U is a neighborhood of x in Y with $y_0 \notin U$ and $n < \omega$.
(iii) Take as a neighborhood base at (y_0, ω) the sets $(U \times (n, \omega]) \cup (F \times \omega)$, where U is a neighborhood of y_0 in Y, $n < \omega$ and $F \in \mathcal{F}$.

Then, X is Hausdorff. To prove this, let $x_0, x_1 \in X$ with $x_0 \neq x_1$. Now, we consider two cases.

Case 1. x_0 (or x_1) $\in Y \times \omega$. It is easy to see that $\{x_0\}$ (or $\{x_1\}$) is a closed set in X. With (i) above, $\{x_0\}$ (or $\{x_1\}$) is a clopen set in X. Hence, x_0 and x_1 are separated by open subsets in X.

Case 2. $x_0, x_1 \in Y \times \{\omega\}$. Since $\bigcap \mathcal{F} = \emptyset$ and Y is Hausdorff, x_0 and x_1 are separated by open subsets in X.

Obviously, Y is homeomorphic to $Y \times \{\omega\}$. Moreover, $A \times \{\omega\}$ and (y_0, ω) cannot be separated by open subsets in X; for, the closure of any neighborhood of (y_0, ω) intersects $A \times \{\omega\}$. Hence, $f^{-1}(\{0\})$ and $f^{-1}([1/2, 1])$ cannot be separated by open subsets in X. By using Lemma 2.1, we have a contradiction. This completes the proof. □

Remark. The above proof actually contains a somewhat simpler proof of the following fact by Gartside and Glyn in [10, Theorem 1]: *Let Y be a Hausdorff space, p a point of Y and A a non-compact closed subset of Y with $p \notin A$. Then, there exists a larger Hausdorff space X containing Y as a closed subspace in which p and A cannot be separated by open subsets.* Assuming regularity of Y, this fact has been proved by Arhangel'skiĭ and Tartir [4]. Moreover, some generalizations to [4] were obtained by Matveev, Pavlov and Tartir [15]. Our construction giving the above fact is motivated by [15, Lemma 3.2].

Corresponding to Theorems 1.1 and 1.2, for the case of regular spaces, we have the following theorem.

Theorem 2.2. *Let Y be a regular space. Then, Y is weakly C-embedded in every larger regular space X if and only if either Y is Lindelöf or for every two disjoint zero-sets of Y at least one of them is compact.*

Proof. The "if" part is easy to see. (In the Lindelöf case modify the standard proof used to show that regular Lindelöf spaces are normal.) To prove the "only if" part, assume that Y is weakly C-embedded in every larger regular space X, and assume on the contrary that Y is non-Lindelöf and let Z_1 and Z_2 be disjoint zero-sets of Y both of which are non-compact. Then, we may assume that $Y - Z_1$ is non-Lindelöf. Since $Y - Z_1$ is the countable union of zero-sets of Y, there exists a non-Lindelöf zero-set Z_3 of Y such $Z_1 \cap Z_3 = \emptyset$. By using [15, Theorem 2.3], there exists a regular space X containing Y such that Z_1 and Z_3 cannot be separated by open subsets in X. But, this is a contradiction. This completes the proof. □

We have another conclusion as follows:

Theorem 2.3. *Let Y be a T_1-space. Then, Y is weakly C-embedded in every larger T_1-space X if and only if every real-valued continuous function on Y is constant.*

Proof. The "if" part is obvious. To prove the "only if" part, assume that Y is weakly C-embedded in every larger T_1-space X, and assume on the contrary that Y admits a non-constant continuous function $f : Y \to [0, 1]$ and two distinct points y_0 and y_1 in Y like in the proof of Theorem 1.2. Let Z be a T_1 and non-T_2-space, and let z_0 and z_1 be distinct points which are not separated by open subsets in Z. Make the quotient space X obtained from $Y \oplus Z$ by identifying each two points y_i and z_i, $i = 0, 1$. Then, X is T_1 and by Lemma 2.1 Y is not weakly C-embedded in X, a contradiction. This completes the proof. □

Remark. (i) The construction given in the above proof is also useful to give other results related to absolute C, C^* or z-embedding in the realm of Hausdorff spaces. We have the following: *A Hausdorff space Y is C (or equivalently, C^* or z)-embedded in every larger Hausdorff space X if and only if every real-valued continuous function on Y is constant.* Indeed, take a regular space A with distinct two points z_0 and z_1 which are not completely separated in A (see [9, p. 119]), and consider the resulting space X obtained from $Y \oplus A$ by identifying each two points y_i and z_i, $i = 0, 1$. Moreover, notice that if in addition a space Y is assumed to be regular, then the space X constructed in the above is also regular. Hence, we have: *A regular space Y is C (or equivalently, C^* or z)-embedded in every larger regular space X if and only if every real-valued continuous function on Y is constant.* The similar results related to absolute C, C^*, z-embedding in the realm of T_1-spaces are also obtained.

(ii) Let $\mathcal{K}$ (respectively $\mathcal{T}_{3\frac{1}{2}}$) be the class of spaces consisting all compact Hausdorff (respectively all Tychonoff) spaces. Let us comment about absolute weak C-embedding in the realm of a class of spaces contained in $\mathcal{T}_{3\frac{1}{2}}$, for example, normal Hausdorff spaces, paracompact Hausdorff spaces, etc. We have the following result: *Let $\mathcal{C}$ be a class of spaces with $\mathcal{K} \subset \mathcal{C} \subset \mathcal{T}_{3\frac{1}{2}}$. Then, a Tychonoff space Y is weakly C-embedded in every larger space X with $X \in \mathcal{C}$ if and only if Y is almost compact or Lindelöf.* For, consider the Stone–Čech compactification of a Tychonoff space X and use Theorem 1.1.

(iii) In Theorem 1.1, "every larger Tychonoff space X" can be replaced by "every larger Tychonoff space X containing Y as a closed subspace" [5]. Similar replacements hold for Theorems 1.2, 2.2 and 2.3. For, we have: *A Hausdorff (respectively Tychonoff, regular or T_1-) space Y is weakly C-embedded in every larger Hausdorff (respectively Tychonoff, regular or T_1-) space X if and only if Y is weakly C-embedded in every larger Hausdorff (respectively Tychonoff, regular or T_1-) space X containing Y as a closed subspace.* Indeed, this follows from the following facts:

(a) a subspace Y of a space X is weakly C-embedded in X if and only if Y is weakly C-embedded in X_Y, where X_Y is the set X with the topology $\{U \cup V\colon U$ is open in X, and $V \subset X - Y\}$ [2],
(b) if X is a Hausdorff (respectively Tychonoff, regular or T_1-) space and Y is a subspace of X, then the space X_Y as was topologized in (a) is a Hausdorff (respectively Tychonoff, regular or T_1-) space containing Y as a closed subspace [9, 5.1.22].

3. Absolute weak P-embedding

Let γ be an infinite cardinal. A subspace Y of a space X is said to be *P^γ-embedded* in X if every γ-separable continuous pseudo-metric on Y can be extended to a continuous pseudo-metric on X. As is known, Y is P^ω-embedded in X if and only if Y is C-embedded in X [1]. A subspace Y of a space X is said to be *P-embedded* in X if Y is P^γ-embedded in X for every γ [1].

A cardinal generalization of weak C-embedding is introduced in [14]: a subspace Y of a space X is said to be *weakly P^γ-embedded* in X if every γ-separable continuous pseudo-metric on Y can be extended to a pseudo-metric on X which is continuous at each point of $Y \times Y$. Note that a subspace Y of a space X is weakly P^ω-embedded in X if and only if Y is weakly C-embedded in X [14]. Also, we say Y is *weakly P-embedded* in X if Y is weakly P^γ-embedded in X for every γ [14].

Motivated by the result due to Alò–Shapiro [1, p. 183] that *a Tychonoff space Y is P-embedded in every larger Tychonoff space X if and only if Y is almost compact* which is a generalization of Fact 1 in Section 1, we obtained in [14, Theorem 3.10] the following theorem that *A Tychonoff space Y is weakly P-embedded in every larger Tychonoff space X if and only if Y is almost compact or Lindelöf*, which is a generalization of Theorem 1.1. Now, we have similar generalization to Theorems 1.2, 2.2 and 2.3. Namely, we have:

Theorem 3.1. *Let Y be a Hausdorff space. Then, Y is weakly P-embedded in every larger Hausdorff space X if and only if either Y is compact or every real-valued continuous function on Y is constant.*

Theorem 3.2. *Let Y be a regular space. Then, Y is weakly P-embedded in every larger regular space X if and only if either Y is Lindelöf or for every two disjoint zero-sets of Y at least one of them is compact.*

Theorem 3.3. *Let Y be a T_1-space. Then, Y is weakly P-embedded in every larger T_1-space X if and only if every real-valued continuous function on Y is constant.*

Let us prove these theorems. First recall from [14] that Y is weakly P^γ-embedded in X if and only if for every disjoint open collection $\{U_\alpha\colon \alpha \in \Omega\}$ of Y with $|\Omega| \leqslant \gamma$ such that $\bigcup_{\alpha\in\Omega} U_\alpha$ is a cozero-set of Y, there exists a disjoint open collection $\{V_\alpha\colon \alpha \in \Omega\}$ of X such that $U_\alpha \subset V_\alpha$ for every $\alpha \in \Omega$. Since weak C-embedding is the same as P^ω-embedding and we have already obtained Theorem 2.2, to prove Theorem 3.2, it suffices to show the following: *Assume that Y is either Lindelöf or for every two disjoint zero-sets of Y at least one of them is compact. Then each disjoint open collection of Y whose union is a cozero-set of Y must be countable.* Notice that this fact and Theorems 1.2 and 2.3 also lead to proofs of Theorems 3.1 and 3.3.

Let $\{U_\alpha\colon \alpha \in \Omega\}$ be a disjoint open collection of Y satisfying that $\bigcup_{\alpha\in\Omega} U_\alpha$ is a cozero-set of Y. Let $f: Y \to [0,1]$ be a continuous function such that $\bigcup_{\alpha\in\Omega} U_\alpha = f^{-1}((0,1])$. For every $n \in \mathbb{N}$, put $\mathcal{U}_n = \{U_\alpha \cap f^{-1}([1/n,1])\colon \alpha \in \Omega\}$.

Case 1. Assume Y is Lindelöf. Then, since $\mathcal{U}_n$ is an open cover of $f^{-1}([1/n,1])$, it follows that $|\mathcal{U}_n| \leqslant \omega$ for every $n \in \mathbb{N}$. Hence, we have $|\Omega| \leqslant \omega$.

Case 2. Assume for every two disjoint zero-sets of Y at least one of them is compact. Then, we shall show that $|\mathcal{U}_n| < \omega$ for every $n \in \mathbb{N}$. To prove this, assume on the contrary that $|\mathcal{U}_n| \geqslant \omega$ for some $n \in \mathbb{N}$. Then, we can put $\mathcal{U}_n = \mathcal{A}_1 \cup \mathcal{A}_2$, where $\mathcal{A}_1 \cap \mathcal{A}_2 = \emptyset$ and $|\mathcal{A}_1| \geqslant \omega$ and $|\mathcal{A}_2| \geqslant \omega$. Notice that $\bigcup \mathcal{A}_1$ and $\bigcup \mathcal{A}_2$ are zero-sets of Y. For, $\bigcup \mathcal{U}'$ is a cozero-set of Y for every $\mathcal{U}' \subset \mathcal{U}$, and $\bigcup \mathcal{A}_i = f^{-1}([1/n,1]) \cap (Y - \bigcup\{U_\alpha\colon U_\alpha \cap f^{-1}([1/n,1]) \notin \mathcal{A}_i\})$, $i = 1,2$. Therefore, $\bigcup \mathcal{A}_1$ and $\bigcup \mathcal{A}_2$ are disjoint non-compact zero-sets of Y, a contradiction. Hence, we have $|\mathcal{U}_n| < \omega$ for every $n \in \mathbb{N}$. Thus, we have $|\Omega| \leqslant \omega$.

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References

[1] R.A. Alò, H.L. Shapiro, Normal Topological Spaces, Cambridge University Press, Cambridge, 1974.
[2] A.V. Arhangel'skiĭ, Relative topological properties and relative topological spaces, Topology Appl. 70 (1996) 87–99.
[3] A.V. Arhangel'skiĭ, From classic topological invariants to relative topological properties, Sci. Math. Japon. 55 (2002) 153–201.
[4] A.V. Arhangel'skiĭ, J. Tartir, A characterization of compactness by relative separation property, Questions Answers Gen. Topology 14 (1996) 49–52.
[5] A. Bella, I.V. Yaschenko, Lindelöf property and absolute embeddings, Proc. Amer. Math. Soc. 127 (1999) 907–913.
[6] R.L. Blair, On υ-embedded sets in topological spaces, in: TOPO '72—General Topology and Its Applications, in: Lecture Notes in Math., Vol. 378, Springer-Verlag, Berlin, 1974, pp. 46–79.
[7] R.L. Blair, A.W. Hager, Extensions of zero-sets and of real-valued functions, Math. Z. 136 (1974) 41–52.

[8] R. Doss, On uniform spaces with a unique structure, Amer. J. Math. 71 (1949) 19–23.
[9] R. Engelking, General Topology, Heldermann, Berlin, 1989.
[10] P.M. Gartside, A. Glyn, Relative separation properties, Topology Appl. 122 (2002) 625–636.
[11] L. Gillman, M. Jerison, Rings of Continuous Functions, Van Nostrand, Princeton, NJ, 1960.
[12] A.W. Hager, D.G. Johnson, A note on certain subalgebras of $C(X)$, Canad. J. Math. 20 (1968) 389–393.
[13] E. Hewitt, A note on extensions of continuous functions, An. Acad. Brasil. Ci. 21 (1949) 175–179.
[14] T. Hoshina, K. Yamazaki, Weak C-embedding and P-embedding, and product spaces, Topology Appl. 125 (2002) 233–247.
[15] M.V. Matveev, O.I. Pavlov, J.K. Tartir, On relatively normal spaces, relatively regular spaces, and on relative property (a), Topology Appl. 93 (1999) 121–129.
[16] Y. Smirnov, Mappings of systems of open sets, Mat. Sb. 31 (1952) 152–166 (in Russian).